Partial Solution Set, Leon Section 4.2

4.2.2a Define $L : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ by $L\left((x_1, x_2, x_3)^T\right) = (x_1 + x_2, 0)^T$. Find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbf{R}^3$.

Solution: We take as the columns of A the images under L of the standard basis vectors from $\mathbf{R}^2$, obtaining $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.2.4 $L : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is given by $L(\mathbf{x}) = (2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2)^T$. The standard matrix representation of L is (again) the matrix A whose ith column is $L(\mathbf{e}_i)$, i.e.,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

4.2.5b L is the operator on $\mathbf{R}^2$ whose effect is to first reflect a vector $\mathbf{x}$ about the x_1 axis and then rotate it counterclockwise through an angle of $\pi/2$. We are to find the standard matrix representation for L.

This can be found by considering the effect of the transformation on the standard basis vectors: $\mathbf{e}_1$ is unaffected by the reflection, and the subsequent rotation produces a copy of $\mathbf{e}_2$, while the reflection sends $\mathbf{e}_2$ to $-\mathbf{e}_2$, and the subsequent rotation sends $-\mathbf{e}_2$ to $\mathbf{e}_1$. The overall effect of L is reflection about the identity line. The matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Note that we can do this in a different (and simpler, if messier) way. It is not hard to show that the matrix representation of the composition of transformations is the product of the individual matrix representations. So let $L = L_2 \circ L_1$, where L_1 is the reflection and L_2 is the rotation. The standard matrix representations for L_1 and L_2 are $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4.2.6 We are given three vectors $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$. The linear transformation L, mapping $\mathbf{R}^2$ to $\mathbf{R}^3$, is given by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3.$$

The problem is to find the matrix A representing L with respect to the bases $[\mathbf{e}_1, \mathbf{e}_2]$ and $B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$. The matrix is A$= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$, where $\mathbf{a}_i = [L(\mathbf{e}_i)]_B$. Thus to find $\mathbf{a}_1$ we first compute $L(\mathbf{e}_1) = \mathbf{b}_1 + \mathbf{b}_3$; the coordinate vector of $\mathbf{b}_1 + \mathbf{b}_3$ with respect to the basis

B is $(1, 0, 1)^T$. Similarly we find $L(\mathbf{e}_2) = \mathbf{b}_2 + \mathbf{b}_3$; the coordinate vector with respect to B is $(0, 1, 1)^T$. So $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Note: In the text, we are given concrete vectors $\mathbf{b}_1 = (1, 1, 0)^T, \mathbf{b}_2 = (1, 0, 1)^T$, and $\mathbf{b}_3 = (0, 1, 1)^T$. But, since the the description of L is in terms of $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$, we need not know what they look like. Had the description been in terms of, say, the standard basis, this would not have been the case.

4.2.8a We have the same vectors $\mathbf{y}_i$ as in problem 7. We are given a linear transformation $L : \mathbf{R}_3 \to \mathbf{R}^3$, where

$$L(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3) = (c_1 + c_2 + c_3)\mathbf{y}_1 + (2c_1 + c_3)\mathbf{y}_2 - (2c_2 + c_3)\mathbf{y}_3.$$

We are to find a matrix A that represents L with respect to the ordered basis $[\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]$. We can do this in at least two ways.

(i) Using corollary 4.2.4: we have $m = n$ and $E = F = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]$. The corollary leads us to perform Gaussian elimination on the augmented matrix

$$[\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3 | L(\mathbf{y}_1) L(\mathbf{y}_2) L(\mathbf{y}_3)] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & -1 & 1 \\ 1 & 1 & 0 & 3 & 1 & 2 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{array} \right].$$

The result of elimination is

$$[I|A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array} \right].$$

(ii) Not using the corollary: We are using the same basis throughout. For that reason, using corollary 4.2.4 actually complicates things. In other words, only when we get to part (b) of this problem does it make any difference what $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ actually look like. The first column of A is $L(\mathbf{y}_1) = \mathbf{y}_1 + 2\mathbf{y}_2$; the coordinates of this sum with respect to Y are $(1, 2, 0)$. The second column of A is $L(\mathbf{y}_2) = \mathbf{y}_1 - 2\mathbf{y}_3$; the coordinates of this sum with respect to Y are $(1, 0, -2)$. The third column of A is $L(\mathbf{y}_3) = \mathbf{y}_1 + \mathbf{y}_2 - \mathbf{y}_3$; the coordinates of this sum with respect to Y are $(1, 1, -1)$.

4.2.8b We are given a selection of vectors in R^3. For each, we are asked to find the coordinate vector $[\cdot]_Y$ with respect to $[\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]$, and then to use the matrix from part (a) to find the image. The first of the given vectors is $\mathbf{x} = (7, 5, 2)^T$. It is a simple matter to set up an augmented matrix $[\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3 | \mathbf{x}]$; the solution is $[\mathbf{x}]_Y = (2, 3, 2)^T$. It follows that

$$[L(\mathbf{x})]_Y = A[\mathbf{x}]_Y = (7, 6, -8)^T.$$

Of course, for those who worked problem (7), this is even simpler:

$$[L(\mathbf{x})]_Y = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -8 \end{bmatrix}.$$

4.2.12a The linear operator L defined by

$$L(p(x)) = p'(x) + p(0)$$

maps P_3 into P_2. Find the matrix representation of L with respect to the ordered bases $[x^2, x, 1]$ and $B = [2, 1 - x]$. Find the coordinates of $L(p(x))$ with respect to the given basis for P_2 if $p(x) = x^2 + 2x - 3$.

Solution: We must first compute $L(p(x))$ for each basis polynomial p, and then express each result with respect to the given ordered basis for P_2.

1. $L(x^2) = 2x + 0^2 = 2x$; $[2x]_B = (1, -2)^T$.
2. $L(x) = 1 + 0 = 1$; $[1]_B = (\frac{1}{2}, 0)^T$.
3. $L(1) = 0 + 1 = 1$; $[1]_B = (\frac{1}{2}, 0)^T$.

It follows that the matrix we want is $A = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix}$. The coordinate vector of $p(x) = x^2 + 2x - 3$ with respect to the given basis for P_3 is $\mathbf{z} = (1, 2, -3)^T$. The coordinate vector of $L(p(x))$ with respect to the basis B is therefore $A\mathbf{z} = (1/2, -2)^T$. Thus $L(x^2 + 2x - 3) = \frac{1}{2}(2) - 2(1 - x) = 2x - 1$, as expected.

4.2.16a We have bases $E = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $F = [\mathbf{b}_1, \mathbf{b}_2]$, where $\mathbf{u}_1 = (1, 0, -1)^T, \mathbf{u}_2 = (1, 2, 1)^T$, $\mathbf{u}_3 = (-1, 1, 1)^T$, $\mathbf{b}_1 = (1, -1)^T$, and $\mathbf{b}_2 = (2, -1)^T$. We are to find the matrix A representing $L(\mathbf{x}) = (x_3, x_1)^T$ with respect to E and F.

Solution: Following the advice in theorem 4.2.3 and its corollary, we begin by finding $L(\mathbf{u}_i)$ for $i = 1, 2, 3$: we find that $L(\mathbf{u}_1) = (-1, 1)^T$, $L(\mathbf{u}_2) = (1, 1)^T$, and $L(\mathbf{u}_3) = (1, -1)^T$. Letting B be the matrix with columns $\mathbf{b}_1$ and $\mathbf{b}_2$, we find $B^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$. For each vector $L(\mathbf{u}_i)$, we calculate $[L(\mathbf{u}_i)]_B = B^{-1}L(\mathbf{u}_i)$ for $i = 1, 2, 3$; the results are the columns of $A = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

4.2.18 Let V, W be vector spaces, with ordered bases E and F, respectively. Let $L : V \to W$ be a linear transformation, with matrix representation (relative to E and F) A. Show:

1. $\mathbf{v} \in \ker(L)$ if and only if $[\mathbf{v}]_E \in N(A)$.

   Proof: $\mathbf{v} \in \ker(L)$ iff $L(\mathbf{v}) = \mathbf{0}$ iff $A[\mathbf{v}]_E = \mathbf{0}$ iff $[\mathbf{v}]_E \in N(A)$. $\qquad\square$

2. $\mathbf{w} \in L(V)$ if and only if $[\mathbf{w}]_F \in CS(A)$.

   Proof: $\mathbf{w} \in L(V)$ iff $L(\mathbf{v}) = \mathbf{w}$ for at least one $\mathbf{v} \in V$ iff $A[\mathbf{v}]_E = [\mathbf{w}]_F$ for at least one $\mathbf{v} \in V$ iff $[\mathbf{w}]_F \in CS(A)$. □

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